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Analysis for a System of Coupled Reaction-Diffusion Parabolic Equations Arising in Biology

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Abstract—We study a system where the equations are coupled by transport and reaction terms. In the general case, we prove local existence and uniqueness of the solution, and in a particular case for *a priori* fixed time T, global existence on $[0,T]$.

Keywords—Nonlinear, Parabolic system, Chemotaxis.

1. INTRODUCTION

1.1. Chemotaxis

In this paper, we investigate a mathematical model proposed by Keller and Segel [1]. Other similar models were set up and analysed by Alt [2] and Schaaf [3]. This model describes the movement of the single-cell amoebae towards the region of relatively high concentration of a chemical called cyclic-AMP, which is produced by the amoebae themselves. This phenomenon is called "chemotaxis."

Let Ω be a smoothly bounded domain of $\mathbb{R}^3$ and a real $T > 0$. Here, we study the existence and uniqueness of the solution of the following nonlinear parabolic system:

$$(A)\quad \begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = -\operatorname{div}(u \nabla \psi(a)), & \text{on }]0,T[\times \Omega, \qquad (1) \\ \dfrac{\partial a}{\partial t} - d_2 \Delta a + k_2 a = k_1 u, & \qquad (2) \\ \dfrac{\partial u}{\partial n} - u \dfrac{\partial \psi(a)}{\partial n} = 0 \text{ and } \dfrac{\partial a}{\partial n} = 0, & \text{on } \Gamma = \partial\Omega, \\ u(0,.) = u_0,\ a(0,.) = a_0, & \text{with given } u_0 \text{ and } a_0 \text{ on } \Omega. \end{cases}$$

These boundary conditions reveal that the population and the attractant flux are zero through Ω border. The variables u and a represent, respectively, the population density and the chemical concentration. In the first equation, the flux is the sum of the chemotaxis flux $u\psi'(a)\nabla a$ and the diffusion flux $d_1 \nabla u$ where d_1 is the diffusion coefficient of u. The second equation shows that the cyclic AMP diffuses with a diffusion coefficient d_2 and is produced by the amoebae. The diffusion coefficients verify $d_2 > d_1 > 0$. The term $k_1 u - k_2 a$ is the reaction term where k_1, k_2 are positive constants. Here $k_1 u$ represents the spontaneous production of the cyclic AMP, and is proportional to the number of amoebae, while $-k_2 a$ represents decay of the attractant activity.

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This article is structured as follows. In Section 2, we prove the local existence and the uniqueness of the solution of (A), then in Section 3 for *a priori* fixed time T, during which we want to observe the phenomenon of chemotaxis, we draw conditions on the initial data, permitting to ensure the global existence of the solution on $[0, T]$. The initial data depend of T. We detail the proof in Section 3.

Jäger and Luckhaus [4] have studied a similar model with $\psi'(a)$, a constant. They have showed, in two space dimensions and radially symmetric situations, the blow up of u in finite time. The global existence of smooth solutions to similar systems was studied by Pozio and Tesei [5].

1.2. Preliminaries

We begin by defining some notations and function spaces which will be used later. We shall use the notations: V for the Hilbert space $H^1(\Omega)$ and $|.|_{0,\Omega}$, $|.|_{0,n,\Omega}$, $|.|_{\infty,\Omega}$, $\|.\|_{m,\Omega}$ for the respective norms in $L^2(\Omega)$, $(L^2(\Omega))^n$, $L^\infty(\Omega)$, and $H^m(\Omega)$, $m \geq 1$. We keep the same notations for the norms on open $Q =]O, T[\times \Omega$. V' is the dual of V, with the well-known situation $V \subset_{\succ} L^2(\Omega) \subset_{\succ} V'$. $\langle .,. \rangle$ is duality V', V. We introduce the Hilbert space $W(0,T)$, $W(0,T) = \{v \in L^2(0,T;V),\ \frac{\partial v}{\partial t} \in L^2(0,T;V')\}$ equipped with the graph norm: $\|v\|_W^2 = \int_0^T \|v\|_{1,\Omega}^2 \, dt + \int_0^T \|\frac{\partial v}{\partial t}\|_{V'}^2 \, dt$. u_0 and a_0 are given in $L^\infty(\Omega)$ with $a_0 \geq 0$ and $u_0 \geq 0$, ψ is Lipschitzian on $\mathbb{R}$.

2. LOCAL EXISTENCE AND UNIQUENESS RESULT

2.1. Variational Problem and Progress of the Work

Thus, we must study the following variational problem:

$$
(P) \quad \begin{cases}
u \in W(0,T) \cap L^\infty(Q), & a \in W(0,T), \\
\forall \varphi \in V, & \text{a.e. } t \in]0,T[, \\
\left\langle \dfrac{\partial u}{\partial t}, \varphi \right\rangle + d_1 \displaystyle\int_\Omega \nabla u \nabla \varphi \, dx = \int_\Omega u \nabla \psi(a) \nabla \varphi \, dx, & (3) \\
\left\langle \dfrac{\partial a}{\partial t}, \varphi \right\rangle + d_2 \displaystyle\int_\Omega \nabla a \nabla \varphi \, dx + k_2 \int_\Omega a \varphi \, dx = k_1 \int_\Omega u \varphi \, dx, & (4) \\
u(0,.) = u_0 \text{ and } a(0,.) = a_0.
\end{cases}
$$

As $W(0,T)$ is included in $C([0,T]; L^2(\Omega))$, then $u(0,.)$ and $a(0,.)$ are well defined.

The writing of the right-hand side of equation (3) set a first difficulty because it is necessary that the product $u \nabla \psi(a)$ should belong to $L^2(Q)$. If $u \in L^\infty(Q)$, this result is ensured but for arbitrary initial data, it is unlikely to have u bounded in Q for an arbitrary $T > 0$. Therefore, we use a truncature method; that is to say, we will introduce the following β function defined over $\mathbb{R} : \beta(u) = u^+ - (u - M)^+$ with M a positive constant.

First we investigate the following new problem:

$$
(P_1) \quad \begin{cases}
u \in W(0,T), & a \in W(0,T), \\
\forall \varphi \in V, & \text{a.e. } t \in]0,T[, \\
\left\langle \dfrac{\partial u}{\partial t}, \varphi \right\rangle + d_1 \displaystyle\int_\Omega \nabla u \nabla \varphi \, dx = \int_\Omega \beta(u) \nabla \psi(a) \nabla \varphi \, dx, \\
\left\langle \dfrac{\partial a}{\partial t}, \varphi \right\rangle + d_2 \displaystyle\int_\Omega \nabla a \nabla \varphi \, dx + k_2 \int_\Omega a \varphi \, dx = k_1 \int_\Omega u \varphi \, dx, \\
u(0,.) = u_0 \text{ and } a(0,.) = a_0.
\end{cases}
$$

We will prove the global existence and the uniqueness of the solution of the problem (P_1) and show that this solution is continuous in $\bar{Q}$. According to this result, by choosing the appropriate M constant, the local existence and uniqueness of the solution of the problem (P) can be inferred.

To study the problem (P_1), we use a fixed point method [6]. We introduce two new problems. First, we must make a classical change of unknown functions (cf. [7]), so as to make V elliptic bilinear forms appear.

Let $\lambda \in \mathbb{R}_+^*$ be given; we set $v = ue^{-\lambda t}$ and $b = ae^{-\lambda t}$. We gain a new problem (P_2) which is equivalent to (P_1). Then we uncouple this new problem as follows: given w in $W(0,T)$, we consider

$$(P_3)\quad \begin{cases} v \in W(0,T), & b \in W(0,T), \\ \forall \varphi \in V, & \text{a.e. } t \in]0,T[, \\ \left\langle \dfrac{\partial v}{\partial t}, \varphi \right\rangle + d_1 \displaystyle\int_\Omega \nabla v \nabla \varphi \, dx = \int_\Omega \beta(w) \nabla \psi(b) \nabla \varphi \, dx, & \\ \left\langle \dfrac{\partial b}{\partial t}, \varphi \right\rangle + d_2 \displaystyle\int_\Omega \nabla b \nabla \varphi \, dx + k_2 \int_\Omega b \varphi \, dx = k_1 \int_\Omega w \varphi \, dx, & \\ v(0,.) = u_0 \text{ and } b(0,.) = a_0. & \end{cases}$$

Using a Lions theorem [8], we prove the existence and the uniqueness of the solution (v, b) of problem (P_3), then applying the standard methods, we establish *a priori* estimates of b, v in $W(0,T)$. According to these results, we can apply the corollary of Schauder Tychonoff's theorem as, for an appropriate choice of λ, there exists a bounded closed convex subset of $W(0,T)$ invariant by the transformation $w \to v$. Therefore, we prove the existence of the solution (u, a) of (P_1). Furthermore, it's easy to show that $\forall t \in [0,T]$ and a.e. $x \in \Omega$, $a(t,x) \geq 0$ and $u(x,t) \geq 0$.

2.2. Uniqueness of the Solution (u, a) for (P_1)

After the results of Ladyzenskaya [9] as to Neumann's linear parabolic problems and results stated in [10], we have the following lemma.

LEMMA 1. *If one assumes $a_0 \in H^2(\Omega)$, such that $\frac{\partial a_0}{\partial n} = 0$ on Γ, then the solution a of (P_1) verifies $a \in L^2(0,T; H^3(\Omega)) \cap C([0,T]; H^2(\Omega))$ and $\frac{\partial a}{\partial t} \in W(0,T)$.*

THEOREM 2. *Consequently, since we work in an open of $\mathbb{R}^3$ according to Sobolev's embedding theorem, we have $\nabla a \in L^2(0,T; (L^\infty(\Omega))^3) \cap C([0,T]; V^3)$.*

In these conditions and supposing ψ' Lipschitzian on all bounded of $\mathbb{R}$, we consider (u, a) a solution of (P_1) which verifies the regular properties of Lemma 1 associated with initial conditions (u_0, a_0) and $(\hat{u}, \hat{a})$ an arbitrary solution of (P_1) associated with initial conditions $(\hat{u}_0, \hat{a}_0)$. Then, subtracting the corresponding identities and using Gronwall's lemma [11], it ensues: $\forall t \in [0,T]$,

$$|(u - \hat{u})(t)|_{0,\Omega}^2 \leq \left(K \left(1 + \|\nabla a\|^2_{L^2(0,T;(L^\infty(\Omega))^n)} \right) |a_0 - \hat{a}_0|^2_{0,\Omega} + |u_0 - \hat{u}_0|^2_{0,\Omega} \right) \exp\left(\int_0^t m(s)\, ds \right),$$

where $m(s) = K(1 + |\nabla \psi(a(s))|^2_{\infty,n,\Omega})$ and K is a positive constant. Thus, we deduce the uniqueness theorem.

THEOREM 3. *If one assumes $a_0 \in H^2(\Omega)$ such that $\frac{\partial a_0}{\partial n} = 0$ on Γ, ψ' Lipschitzian on all bounded of $\mathbb{R}$, the problem (P_1) has a unique solution.*

2.3. Continuity of the Solution for Variational Problem (P_1)

According to the above results and Ladyzenskaya's results, we have the following lemma.

LEMMA 4. *If one assumes $u_0 \in H^2(\Omega)$, such that $\frac{\partial u_0}{\partial n} = 0$ on Γ, and $a_0 \in H^3(\Omega)$, such that $\frac{\partial a_0}{\partial n} = 0$ on Γ, the solution (u, a) of (P_1) verifies the following properties:*

$$a \in L^\infty\left(0,T; H^3(\Omega)\right), \quad \frac{\partial a}{\partial t} \in C([0,T]; V) \cap L^2\left(0,T; H^2(\Omega)\right), \quad \frac{\partial^2 a}{\partial t^2} \in L^2(Q), \quad \text{and}$$
$$u \in C_s\left([0,T]; H^2(\Omega)\right), \quad \frac{\partial u}{\partial t} \in L^2(0,T;V) \cap C\left([0,T]; L^2(\Omega)\right).$$

Here $C_s([0,T];H^2(\Omega))$ is the space of continuous functions from $[0,T]$ to $H^2(\Omega)$ equipped with its weak topology. To prove these results, we use Faedo-Galerkin's method [12] and the property $C([0,T];V)\cap L^\infty(0,T;H^2(\Omega)) = C_s([0,T];H^2(\Omega))$ (see [13]).

2.4. Existence and Uniqueness Result for Problem (P)

THEOREM 5. *Let us suppose $a_0 \in H^3(\Omega)$, such that $\frac{\partial a_0}{\partial n} = 0$ on Γ and $u_0 \in H^2(\Omega)$, such that $\frac{\partial u_0}{\partial n} = 0$ on Γ, then there exists at least a time t_0, $t_0 > 0$, for which the problem (P) has a unique solution (u,a) on $[0,t_0]\times\Omega$ with $u \in C_s([0,t_0];H^2(\Omega))$ and $a \in C([0,t_0];H^2(\Omega))$. Furthermore, this solution verifies $\frac{\partial a}{\partial t} \in W(0,T)$ and $a \in L^2(0,T;H^3(\Omega))$.*

PROOF. We can take $\beta(u) = u^+ - (u-M)^+$ for β function in (P_1) problem with $M = 1 + \|u_0\|_{\infty,\Omega}$. By the above work for all $T > 0$, we know that the problem (P_1) has a unique solution (u,a) with $a \in C([0,T];H^2(\Omega))$ and $u \in C_s([0,T];H^2(\Omega))$. But for all $\alpha > 0$, $H^2(\Omega) \subset_{\succ\succ} H^{2-\alpha}(\Omega)$, with compact injection, it follows that $u \in C([0,T];H^{2-\alpha}(\Omega))$. Since the function $u : t \to u(t,.)$ is continuous from $[0,T]$ to $H^{2-\alpha}(\Omega)$, we have

$$\forall\varepsilon > 0, \qquad \exists t_0 \in [0,T]\,/\forall t, t \le t_0 \Rightarrow \|u(t,.) - u_0\|_{2-\alpha,\Omega} \le \varepsilon.$$

Consequently, if we choose α such that $2-\alpha > 3/2$, then

$$\exists t_0 \in [0,T]\,/\forall t,\ t \le t_0 \Rightarrow |u(t,.)|_{\infty,\Omega} \le 1 + |u_0|_{\infty,\Omega}.$$

Finally, $\forall t \in [0,t_0]$, $\beta(u) = u$, hence, the (P) problem has a unique solution (u,a) on $[0,t_0]\times\Omega$.

3. GLOBAL EXISTENCE ON $[0,T]$, T FIXED AND UNIQUENESS OF THE SOLUTION FOR PROBLEM (P)

With intention of global existence, we want to show with particular initial conditions that $t^* = \sup\{t \in [0,T]/(P)$ has a unique solution on $[0,t]\times\Omega\}$ verifies $t^* = T$. For that, assuming $t^* < T$ if we can prolong our solution in t^* and show that the solution in t^* has the same regularity as u_0 and a_0, we will be able to repeat the preceding reasoning and show the contradiction. Therefore, it is necessary to show that u is bounded in $]0,t_0[\times\Omega$ by a constant independent of t_0. But after the previous work, we know that if u is bounded in $W(0,T)$, then u is bounded in $L^\infty(Q)$.

THEOREM 6. *For all $T > 0$, there exists a constant γ_T, such that if the initial data a_0 and u_0 verify $\|a_0\|_{2,\Omega} < \gamma_T$, $|u_0|_{\infty,\Omega} < \gamma_T$, and $|\nabla u_0|_{0,n,\Omega} < \gamma_T$, then the problem (P) has a unique solution on $[0,T]\times\Omega$.*

REMARK. $\lim_{T\to+\infty}\gamma_T = 0$ so Theorem 5 does not prove the existence of the solution for all $t \in]0,+\infty[$ and it is not out of the question that appears a blow-up phenomenon in finite time.

PROOF.

FIRST STEP. We show that the local solution (u,a) of (P) found in Section 2.4, defined on $[0,t_0]\times\Omega$ is bounded on $[0,t_0]\times\Omega$ by a constant independent of time t_0.

(a) u bounded in $L^\infty(0,t_0;L^2(\Omega))\cap L^2(0,t_0;V)$.

We will denote through all this paragraph by R any constant depending only of the coefficients Ω, T and independent of the initial data and t_0.

We have shown in Section 2.4 that $u \in C_s([0,t_0];H^2(\Omega))$, $a \in C([0,t_0],H^2(\Omega))$, and $\frac{\partial a}{\partial t} \in L^2(0,t_0;H^2(\Omega))\cap L^\infty(0,t_0;V)$. As a result, this regularity justifies the calculations below. We multiply the equality (2) by $-\Delta a$, we find that for $t \in]0,t_0[$

$$|\Delta a|^2_{0,Qt} \le R\,\|a_0\|_{2,\Omega} + R\int_0^t \|u\|^2_V\,ds,$$

$$\|a\|^2_{L^\infty(0,t;V)} \le R\,\|a_0\|_{2,\Omega} + R\int_0^t \|u\|^2_V\,ds, \tag{5}$$

$$\|\nabla a\|^2_{L^2(0,t;(L^2(\Omega))^n))} \le |\nabla a_0|^2_{0,n,\Omega}\left(e^t - 1\right) + k_1^2\int_0^t\left(\int_0^s\int_\Omega |\nabla u|^2\,dx\,d\tau\right)e^s\,ds. \tag{6}$$

Afterwards, as $\frac{\partial a}{\partial t} \in L^2(0,T;H^2(\Omega))$, we can multiply (2) by $-\Delta\frac{\partial a}{\partial t}$, it ensues

$$\|\Delta a\|^2_{L^\infty(0,t;L^2(\Omega))} \leq R\,\|a_0\|_{2,\Omega} + R\|u\|^2_{L^2(0,t;V)}, \tag{7}$$
$$\left\|\frac{\partial a}{\partial t}\right\|^2_{L^2(0,t;V)} \leq R\,\|a_0\|_{2,\Omega} + R\|u\|^2_{L^2(0,t;V)}.$$

Moreover (cf. [10]), we have

$$\|a\|^2_{L^2(0,t;H^3(\Omega))} \leq R\,\|a_0\|_{2,\Omega} + R\|u\|^2_{L^2(0,t;V)}. \tag{8}$$

Now in equation (3), we choose $\varphi = u - u_0$. After noticing

$$d_1 \int_0^t \int_\Omega \nabla u\,(\nabla u - \nabla u_0)\,dx\,ds \leq \frac{d_1}{2}\int_0^t\int_\Omega |\nabla u|^2\,dx\,ds + \frac{d_1}{2}\int_0^t\int_\Omega |\nabla(u-u_0)|^2\,dx\,ds$$
$$- \frac{d_1}{2}\int_0^t\int_\Omega |\nabla u_0|^2\,dx\,ds,$$

it results that the quantity $Eq(t) = \int_\Omega (u(t)-u_0)^2\,dx + d_1\int_0^t\int_\Omega |\nabla u|^2\,dx\,ds$ verifies according to (6) and Gronwall's lemma [11]:

$$Eq(t) \leq \left\{ d_1 t\,|\nabla u_0|^2_{0,n,\Omega} + \frac{2\,|u_0|^2_{\infty,\Omega}\,|\psi'|^2_{\infty,\mathbb{R}}}{d_1}\,|\nabla a_0|^2_{0,n,\Omega}\left(e^t - 1\right)\right\} e^{\left(\left(2|\psi'|^2_{\infty,\mathbb{R}}\right)/(d_1)\right)\int_0^t \widetilde{m}(s)\,ds}, \tag{9}$$

with

$$\widetilde{m}(s) = \max\left(m(s), \frac{|u_0|^2_{\infty,\Omega}\,k_1^2}{d_1}e^s\right),$$

where $m(s) = |\nabla a(s)|^2_{\infty,n,\Omega}$.

If we set

$$y(t) = \int_0^t\int_\Omega (u(s)-u_0)^2\,dx\,ds + d_1\int_0^t\int_\Omega |\nabla u|^2\,dx\,ds,$$

then according to (8), it can be inferred

$$\frac{2\,\|\psi'\|^2_{\infty,\mathbb{R}}}{d_1}\int_0^t \widetilde{m}(s)\,ds \leq \frac{2\,|\psi'|^2_{\infty,\mathbb{R}}}{d_1}\left(R\,\|a_0\|_{2,\Omega} + R\|u\|^2_{L^2(0,t;V)} + \frac{|u_0|^2_{\infty,\Omega}\,k_1^2}{d_1}t\right)$$
$$\leq R\,\|a_0\|_{2,\Omega} + Ry(t) + R\,|u_0|^2_{\infty,\Omega}\,t. \tag{10}$$

With this result, we obtain

$$R\,y(t)e^{-R\,y(t)} \leq R(1+T)K\,(a_0,\nabla u_0)\,e^{C(a_0,u_0)},$$

with

$$K\,(a_0,\nabla u_0) = d_1 T\,|\nabla u_0|^2_{0,n,\Omega} + \frac{2\,|u_0|^2_{\infty,\Omega}\,|\psi'|^2_{\infty,\mathbb{R}}}{d_1}\,|\nabla a_0|^2_{0,n,\Omega}\left(e^T - 1\right)$$

and $C(a_0,u_0) = R|a_0|^2_{2,\Omega} + R|u_0|^2_{\infty,\Omega}T$.

Now the function $u \to ue^{-u}$ is strictly increasing and strictly bounded for above by $1/e$ on $[0,1[$. Thereby, let $T > 0$, if initial data a_0 and u_0 are such that $e^{C(a_0,u_0)}K(a_0,\nabla u_0) < 1/(R(1+T)e)$, as $y(0) = 0$ and y is continuous in $[0,T]$, we obtain $y(t) < 1/R$ on $[0,\min(t_0,T)]$. Hence if, for example, the initial data a_0 and u_0 verify $\|a_0\|_{2,\Omega} < \gamma_T$, $|u_0|_\infty < \gamma_T$ and $|\nabla u_0|_{0,n,\Omega} < \gamma_T$, with the constant γ_T, such that

$$e^{R\gamma_T^2(1+T)}\left(d_1 T\gamma_T^2 + \frac{2\gamma_T^4\,|\psi'|^2_{\infty,\mathbb{R}}\left(e^T-1\right)}{d_1}\right) \leq \frac{1}{R(1+T)e},$$

it follows from the definition of $y(t)$ that u is bounded in $L^2(0,t_0;V)$, then according to (9) in $L^\infty(0,t_0;L^2(\Omega))$. Furthermore, we can deduce the following estimates from inequalities (9) and (10):

$$\begin{aligned} \|u-u_0\|^2_{L^\infty(0,T;L^2(\Omega))} &\le K\left(a_0,\nabla u_0\right)e^{C(a_0,u_0)+1},\\ \|u\|^2_{L^2(0,T;V)} &\le K\left(a_0,\nabla u_0\right)e^{C(a_0,u_0)+1}. \end{aligned} \tag{11}$$

(b) u bounded in $L^\infty(0,t_0;H^2(\Omega))$ by a constant independent of t_0.

We denote here by C any constant independent of t_0. We multiply the equality (1) by $-\Delta u$, according to the injection continuous from V to $L^6(\Omega)$ it comes, with $m(s)=|\nabla a(s,.)|^2_{\infty,n,\Omega}$:

$$\begin{aligned} \int_\Omega |\nabla u|^2\,dx + d_1\int_0^t\int_\Omega |\Delta u|^2\,dx\,ds \le{}& \frac{3}{d_1}\,|\psi'|^2_{\infty,\mathbb{R}}\int_0^t m(s)\int_\Omega |\nabla u|^2\,dx\,ds\\ &+\frac{3}{d_1}\,|\psi'|^2_{\infty,\mathbb{R}}\int_0^t \|\Delta a\|^2_{4,\Omega}\|u\|^2_{4,\Omega}\,ds\\ &+\frac{3}{d_1}\,|\psi''|^2_{\infty,\mathbb{R}}\,\|\nabla a\|^4_{L^\infty(0,t;V^n)}\|u\|^2_{L^2(0,t;V)}+|\nabla u_0|^2_{0,n,\Omega}\,. \end{aligned} \tag{12}$$

From (7),(11) and the results that we can find in [10], we see that exists C depending on initial data a_0 such that

$$\frac{3}{d_1}\,|\psi''|^2_{\infty,\mathbb{R}}\,\|\nabla a\|^4_{L^\infty(0,t;V^n)}\|u\|^2_{L^2(0,t;V)}+|\nabla u_0|^2_{0,n,\Omega}\le C.$$

What is more, from (8) and the inclusion $V\subset_{\succ} L^4(\Omega)$, there exists a constant C such that

$$\frac{3}{d_1}\,|\psi''|^2_{\infty,\mathbb{R}}\int_0^t \|\Delta a\|^2_{4,\Omega}\|u\|^2_{4,\Omega}\,ds \le \int_0^t r(s)\int_\Omega |\nabla u|^2\,dx\,ds + C,$$

with $r(s)=\|\Delta a(s,.)\|^2_{4,\Omega}$, $r\in L^1(]0,t_0[)$.

Using Gronwall's lemma [11], then (10) the inequality (12) is implies

$$\exists C>0,\ \int_\Omega |\nabla u|^2\,dx + \frac{d_1}{2}\int_0^t\int_\Omega |\Delta u|^2\,dx\,ds\le C. \tag{13}$$

Finally, according to the latter inequality we conclude that u is bounded in $L^2(0,t_0;H^2(\Omega))\cap L^\infty(0,t_0;V)$. Moreover, if we choose $\frac{\partial u}{\partial t}$ in (3) for test function, it ensues

$$\begin{aligned} \int_0^t\int_\Omega\left(\frac{\partial u}{\partial t}\right)^2dx\,ds \le{}& 3\,|\psi'|^2_{\infty,\mathbb{R}}\,\|u\|^2_{L^\infty(0,t;V))}\|a\|^2_{L^2(0,t;H^3(\Omega)))}\\ &+3\,|\psi'|^2_{\infty,\mathbb{R}}\,\|u\|^2_{L^2(0,t;H^2(\Omega)))}\|\Delta a\|^2_{L^\infty(0,t;L^2(\Omega))}\\ &+3\,|\psi''|^2_{\infty,\mathbb{R}}\,\|u\|^2_{L^2(0,t;V)}\|\nabla a\|^4_{L^\infty(0,t;V^n)}+d_1\,|\nabla a_0|^2_{0,n,\Omega}\,. \end{aligned}$$

From the results (7),(8),(11), and (13), we may deduce that $\frac{\partial u}{\partial t}$ is bounded in $L^2(Q_{t_0})$ by a constant independent of t_0. Consequently and according to the proof of Lemma 2, a and $\frac{\partial a}{\partial t}$ are bounded, respectively, in $L^\infty(0,t_0;H^3(\Omega))$ and in $L^\infty(0,t_0;V)$ by a constant independent of t_0. Eventually it follows from the latter smoothness results that $\frac{\partial u}{\partial t}$ and u are bounded, respectively, in $L^\infty(0,t;L^2(\Omega))$ and in $L^\infty(0,t_0;H^2(\Omega))$ by a constant independent of t_0 since $\frac{\partial(\beta(u)\nabla\psi(a))}{\partial t}$ is bounded in $L^2(Q_{t_0})$ by a constant independent of t_0.

SECOND STEP. After Theorem 2, we can define an integer $t^*>0$ such that $t^*=\sup\{t\in[0,T]/(P)$ has a unique solution on $[0,t]\times\Omega\}$. We assume that $t^*<T$. But according to the

first step, the solution of problem (P) defined over $[0, t^*[$ is such that u and a are, respectively, bounded in $L^\infty(0, t_0; H^2(\Omega))$ and in $L^\infty(0, t_0; H^3(\Omega))$. Now, as the maps $t \to u(t, .)$ and $t \to a(t, .)$ are, respectively, bounded from $[0, t^*[$ to $H^2(\Omega)$ and $H^3(\Omega)$, in both cases we can extract a subsequence which $*$ weakly converges in $L^\infty(0, t_0; H^2(\Omega))$ and in $L^\infty(0, t_0; H^3(\Omega))$ when $t \to t^*$. As a consequence, it is possible to prolong the functions a and u in t^*, and thus, to define the traces $u(t^*_-, .)$ and $a(t^*_-, .)$ which have the same regularity as u_0 and a_0. With a similar analysis, if we choose $u(t^*_-, .)$ and $a(t^*_-, .)$ as initial data, after Theorem 3, we can prove that exists $t_0 > t^*$ such that (P) has a unique solution (u, a) in $[t^*, t_0[\times \Omega$. Therefore, we have demonstrated the existence and uniqueness of the solution of problem (P) in $[0, t_0] \times \Omega$ with $t_0 > t^*$, which leads us to a contradiction with the definition of t^*. Finally under the initial conditions stated in the first step (a), the problem (P) has a unique global solution in $[0, T]$.

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